Problem Set IV: due Last Class – Hard Deadline

1) This problem asks you to explore the Current Convective Instability (CCI) in a homogeneous medium and its sheared field relative, the Rippling Instability.

a) Consider first a current carrying plasma in a straight magnetic field $\underline{B} = B_0\hat{z}$ - i.e. ignore the poloidal field, etc. Noting that the resistivity η is a function of temperature (ala' Spitzer - c.f. Kulsrud 8.7), calculate the electrostatic resistive instability growth rate, assuming T evolves according to:

$$\frac{\partial T}{\partial t} + \underline{v}\cdot\underline{\nabla}T - \chi_{\parallel}\partial_z^2 T - \chi_{\perp}\nabla_{\perp}^2 T = 0$$

and the electrostatic Ohm's Law is just

$$-\partial_z\phi = \frac{1}{\eta}\frac{d\eta}{dT}\tilde{T}J_0.$$

b) *Thoroughly* discuss the physics of this simple instability, i.e.

- what is the free energy source?
- what is the mechanism?
- what are the dampings and how do they restrict the unstable spectrum?
- how does spectral asymmetry enter?
- what is the cell structure?

c) Use quasilinear theory and the wave breaking limit to estimate the heat flux from the C.C.I.

d) Now, consider the instability in a *sheared* magnetic field.

i.) What difficulties enter the analysis?

ii.) Resolve the difficulty by considering coupled evolution of vorticity, Ohm's Law (in electrostatic limit but with temperature fluctuations) and electron temperature. Compute the growth rate in the limit $\chi_{\parallel}$, $\chi_{\perp} \rightarrow 0$. Compute the mode width. Discuss how asymmetry enters here. Explain why.

e) Noting that $\chi_{\|} >> \chi_{\perp}$ (why? - see Kulsrud 8.7), estimate when parallel thermal conduction becomes an important damping effect. Can $\chi_{\|}$ alone ever absolutely stabilize the rippling mode?

f) Calculate the quasilinear heat flux and use the breaking limit to estimate its magnitude.

2) *Taylor in Flatland*

Taylor awakes one morning, and finds himself in Flatland, a 2D world. Seeking to relax, he sets about reformulating his theory for that planar universe.

a) Write down the visco-resistive 2D MHD equations, and show that *three* quadratic quantities are conserved, as $\eta \to 0$, $\nu \to 0$.

b) Which of these is the most likely to constrain magnetic relaxation? Argue that

  i.) the local version of this quantity is conserved for an 'flux circle', as $\eta \to 0$,

  ii.) the global version is the most "rugged", for finite η.

c) Formulate a 2D Taylor Hypothesis - i.e. that magnetic energy is minimized while the quantity you identified from b.) ii.) is conserved. What equation describes this state? Show that the solution is force-free. What quantity is constant in Flatland? Hence, what is the endstate of Taylor relaxation in 2D?

d) Consider the possibility that $\nu >> \eta$ in Flatland. Derive the mean field evolution equation for mean magnetic potential. Discuss!

e) *Optional - Extra Credit* - Describe the visit of the Terrifying Torus to Flatland. How would 2D Taylor perceive this apparition?

N.B. You may find it useful to consult *Flatland*, by E. Abbott.

3) Reformulate the Sweet–Parker Reconnection problem for weak collisionality. Assume a uniform, strong guide field $B_o\hat{z}$ orthogonal to the plane of reconnection. What can be said about the reconnection speed? [Note: This is an open-ended problem that asks you to synthesize the stories of the current-driven ion–acoustic instability and the resulting scattering of momentum with the S–P problem. You may find it useful to consult relevant parts of Kulsrud, Chapter 14.]

4) a) Show that for incompressible MHD in two dimensions, the basic equations can be written as:

$$(\partial_t + \underline{v}\cdot\nabla)\nabla^2\phi = (B\cdot\nabla)\nabla^2 A + \nu\nabla^2\nabla^2\phi + \tilde{f}$$

$$(\partial_t + \underline{v}\cdot\nabla)A = \eta\nabla^2 A.$$

Here ν is viscosity, η is resistivity, $\underline{v} = \underline{\nabla}\phi\times\hat{z}$ and $\underline{B} = \underline{\nabla}A\times\hat{z}$. $\tilde{f}$ is a random force. Take $P = P(\rho)$.

b) Take $\underline{B} = B_0\hat{x}$ to be a weak in-plane magnetic field. Calculate the real frequency and damping for Alfven waves.

c) Using quasilinear theory, calculate the turbulent resistivity induced by a spectrum of Alfven waves in 2D MHD. For $\nu\rightarrow 0$, interpret your result in terms of the freezing-in-law. Why does viscosity enter your result for part (i)? Why does η enter? Contrast these.

d) Taking $\underline{B} = B_0\hat{x}$ and $\langle\tilde{V}_y\tilde{A}\rangle = -\eta_T\partial A_0/\partial y$ as a definition of turbulent resistivity η_T. Show that at stationarity

$$\eta_T = \eta\langle\tilde{B}^2\rangle/B_0^2,$$

assuming the system has periodic boundary conditions. Discuss your result and its implications. This is a famous result, referred to as the Zeldovich Theorem, after Ya.B. Zeldovich.

e) What happens if one pair of boundaries are open? (Hint: Consider flux thru surface.)

5) a) Derive reduced MHD by two different methods. Explain the physics.

b) What linear waves does reduced MHD support? What happened to the others – i.e. how does the ordering eliminate them? (N.B. It may be useful to read Strauss, '76).

c) Recover 2D MHD from reduced MHD.

d) What are the conservation laws of reduced and 2D MHD?

e) Now, derive the reduced MHD equations when $\underline{B}_o = B_o\,\hat{z}$ and gravity is present, i.e. $\underline{g} = g\,\hat{x}$.

6) Kulsrud; Chapter 11, Problem 1. Ignore the last paragraph.

7) a) Derive the Hasegawa-Wakatani equations. Do this like the derivation for reduced MHD, but:

i) neglect inductive effects, and all magnetic perturbations

ii) retain electron pressure in the Ohm's Law

iii) take electrons isothermal

iv) derive an equation for electron density, including parallel electron flow.

b) Discuss the conservation properties for this system.

c) Derive the quasi-linear equations for the H-W system. What do they mean?

d) Derive the mean vorticity and particle flux.

f) Relate the vorticity flux to the Reynolds stress.

8) Consider a magnetized, incompressible fluid (MHD system) in a strong magnetic field $B_0\hat{z}$. The system (box) has width w and length L, density ρ_0, $\nu \sim \eta$, and forcing such that $R_e \gg 1$. The system is stirred to produce dissipation rate $\in$.

a) Assuming $Ku < 1$, calculate the energy spectrum of the Alfven wave turbulence. Describe the key physics.

b) Determine the maximum value of $\in$ for which the result above can reasonably be expected to be valid.

c) Calling the result in (b) $\in_{crit}$, how does the spectrum behave for $\in \gg \in_{crit}$?